GEOMETRIC MAGIC SQUARES AND CUBES.

The term "geometric" has been applied to that class of magic squares wherein the numbers in the different rows, columns, and diagonals being multiplied together give similar products. They are analogous in all respects to arithmetical magic squares.

Any feature produced in an arithmetical square can likewise be produced in a geometric square, the only difference being that the features of the former are shown by summations while those of the latter are shown by products. Where we use an arithmetical series for one, we use a geometric series for the other, and where one is constructed by a method of differences the other is constructed by ratios.

These geometric squares may be considered unattractive because of the large numbers involved, but they are interesting to study, even though the actual squares are not constructed. The absurdity of constructing large geometric squares can be easily shown. For example, suppose we were to construct an 8th order square using the series 2^0, 2^1, 2^2, 2^3,.... 2^{63}, the lowest number would be 1 and the highest number would be 9,223,372,036,854,775,808. Who would be willing to test the accuracy of such a square by multiplying together the numbers in any of its rows or columns?

Analogous to the arithmetical squares the geometric squares may be constructed with a straight geometric series, a broken geometric series, or a series which has no regular progression.

I have divided the methods of construction into four groups, namely: the "Exponential method," the "Exponential La Hirian method," the "Ratio method," and the "Factorial method."

THE EXPONENTIAL METHOD.

The most common way of constructing these squares is with a straight geometric series, arranged in the same order as a straight

7	0	5
2	4	6
3	8	1

Fig. 1.

2^7	2^0	2^5
2^2	2^4	2^6
2^3	2^8	2^1

Fig. 2.

128	1	32
4	16	64
8	256	2

P = 4096

Fig. 3.

arithmetical series would be in any summation square. This is equivalent to the following.

Form any magic with a straight arithmetical series as in Fig. 1.

Consider these numbers as exponents by repeating any number (in this case 2) before each of them, which will give us a square as shown in Fig. 2. It may be noticed that 2 is taken 12 times as a factor in each of the rows, columns, and diagonals, therefore forming a geometric square with constant products of 4096. The square transposed in natural numbers is shown in Fig. 3.

4	-3	2
-1	1	3
0	5	-2

Fig. 4.

3^4	3^{-3}	3^2
3^{-1}	3^1	3^3
3^0	3^5	3^{-2}

Fig. 5.

81	$\frac{1}{27}$	9
$\frac{1}{3}$	3	27
1	243	$\frac{1}{9}$

Fig. 6.

P = 27

$3\frac{1}{2}$	0	$2\frac{1}{2}$
1	2	3
$1\frac{1}{2}$	4	$\frac{1}{2}$

Fig. 7.

$4^{\frac{7}{2}}$	4^0	$4^{\frac{5}{2}}$
4^1	4^2	4^3
$4^{\frac{3}{2}}$	4^4	$4^{\frac{1}{2}}$

Fig. 8.

128	1	32
4	16	64
8	256	2

Fig. 9.

P = 4096

Figs. 4, 5, and 6 show the same process involving negative exponents.

Figs. 7, 8 and 9 show how fractional exponents may be used; and the use of both fractional and negative exponents is shown in Figs. 10, 11 and 12.

$2\frac{1}{2}$	-1	$1\frac{1}{2}$
0	1	2
$\frac{1}{2}$	3	$-\frac{1}{2}$

Fig. 10.

$3^{\frac{5}{2}}$	3^{-1}	$3^{\frac{3}{2}}$
3^0	3^1	3^2
$3^{\frac{1}{2}}$	3^3	$3^{-\frac{1}{2}}$

Fig. 11.

$\sqrt{243}$	$\frac{1}{3}$	$\sqrt{27}$
1	3	9
$\sqrt{3}$	27	$\sqrt{\frac{1}{3}}$

Fig. 12.

P = 27

2^{-5}	2^9	2^8	2^{-2}
2^6	2^0	2^1	2^3
2^2	2^4	2^5	2^{-1}
2^7	2^{-3}	2^{-4}	2^{10}

Fig. 13.

$\frac{1}{32}$	512	256	$\frac{1}{4}$
64	1	2	8
4	16	32	$\frac{1}{2}$
128	$\frac{1}{8}$	$\frac{1}{16}$	1024

Fig. 14.

P = 1024

Figs. 13 and 14 show the exponential method applied to a fourth order square. The exponents in Fig. 13 taken alone, obviously form an arithmetical magic.

This square is an associated square,[1] with the products of each complementary pair equaling 32.

THE EXPONENTIAL LA HIRIAN METHOD.

Two primary squares are shown in Figs. 15 and 16. One is filled with the powers 0, 1 and 2 of the factor 2, and the other with the powers 0, 1 and 2 of the factor 5. Each primary square in itself

2^0	2^2	2^1
2^2	2^1	2^0
2^1	2^0	2^2

Fig. 15.

5^1	5^2	5^0
5^0	5^1	5^2
5^2	5^0	5^1

Fig. 16.

P = 1000

5	100	2
4	10	25
50	1	20

Fig. 17.

is a geometric magic with triplicate numbers. Figs. 15 and 16 multiplied together, cell by cell, will produce the magic shown in Fig, 17.

The factor numbers in this case, 2 and 5, are not necessarily different, but when they are alike the exponents must suit the condition, to avoid duplicate numbers in the final square. To make this clearer: if we form two primary squares that will add together and

3^0	3^1	3^2	3^3
3^3	3^2	3^1	3^0
3^1	3^0	3^3	3^2
3^2	3^3	3^0	3^1

Fig. 18.

2^0	2^6	2^2	2^4
2^2	2^4	2^0	2^6
2^4	2^2	2^6	2^0
2^6	2^0	2^4	2^2

Fig. 19.

P = 514944

1	192	36	432
108	144	3	64
48	4	1728	9
596	27	16	12

Fig. 20.

form an arithmetical magic, the same factor number may be added to each of these primary squares, using the former numbers as exponents, and the two will become geometric primary squares that will multiply together and form a geometric magic without duplicate numbers.

Figs. 18, 19 and 20 show the same method applied to the fourth

[1] A square is associated when any two numbers opposite to and equidistant from the center of the square, give a constant product in geometric squares, or a constant sum in arithmetical squares. Such pairs are called complementaries.

order squares. This is a Jaina square, and is consequently pandiagonal and also contains the other Jaina features.[2]

Figs. 21, 22, 23 show the application of a double set of factors

3^0	5^1	5^2	3^1
3^1	5^2	5^1	3^0
5^1	3^0	3^1	5^2
5^2	3^1	3^0	5^1

Fig. 21.

2^1	2^0	2^2	7^1
2^2	7^1	2^1	2^0
7^1	2^2	2^0	2^1
2^0	2^1	7^1	2^2

Fig. 22.

P = 21000

2	5	100	21
12	175	10	1
35	4	3	50
25	6	7	20

Fig. 23.

to the primary squares. The constants of Fig. 21 are 3×5^3 and those of Fig. 22 are $2^3 \times 7$. This is also a Jaina square.

THE RATIO METHOD.

If we fill a square with numbers as in Fig. 24, such that the ratios between all horizontally adjacent cells are equal, and the ratios between all vertically adjacent cells are equal, we have a natural square which can be formed into a geometric magic by any of the well-known methods.

The horizontal ratios in Fig. 24 are 2 as represented by the figure at the end of the division line, and the vertical ratios are 3

		2	2
	1	2	4
3	3	6	12
3	9	18	36

Fig. 24.

18	1	12
4	6	9
3	36	2

Fig. 25.

P = 216

as indicated, and Fig. 25 shows the magic arrangement of this series.

In a fourth order square, as in Fig. 26, the horizontal ratios are not necessarily equal, and neither are the vertical ratios. A magic may be made from this natural square by forming the numbers in the upper row into a primary square as in Fig. 27. The numbers in the left-hand column are then formed into another pri-

[2] See reference to the Jaina Square in *Magic Squares and Cubes* by W. S. Andrews.

mary square as in Fig. 28. These two primary squares will then produce the magic shown in Fig. 29.

P = 7560

1	2	3	9
4	8	12	36
5	10	15	45
7	14	21	63

Fig. 26.

1	2	3	9
9	3	2	1
2	1	9	3
3	9	1	2

Fig. 27.

1	7	4	5
4	5	1	7
5	4	7	1
7	1	5	4

Fig. 28.

1	14	12	45
36	15	2	7
10	4	63	3
21	9	5	8

Fig. 29.

Fig. 30 is a balanced natural square. This series will produce a perfect Jaina, a Nasik,[3] or an associated square. Figs. 31, 32 and 33 show it arranged in a Nasik formation.

P = 14400

1	2	5	10
3	6	15	30
4	8	20	40
12	24	60	120

Fig. 30.

1	2	10	5
10	5	1	2
1	2	10	5
10	5	1	2

Fig. 31.

1	12	1	12
3	4	3	4
12	1	12	1
4	3	4	3

Fig. 32.

1	24	10	60
30	20	3	8
12	2	120	5
40	15	4	6

Fig. 33.

Mr. L. S. Frierson's arithmetical equation squares also have their geometric brothers. Where he applies the equation $a - b = c - d$, we use the proportion $a : b :: c : d$. Fig. 35 shows a natural

	A	B	
C	F	E	C
D	E	F	D
	A	B	

Fig. 34.

2	3	4	6
1	7	8	56
28	21	12	9
14	49	24	84

Fig. 35.

84	3	4	14
1	12	21	56
28	8	7	9
6	49	24	2

Fig. 36.

P = 14112

equation square, and besides the proportions there shown, the diagonals of the magic depend on the necessary proportion $a : b :: c : d$ as indicated in the respective cells of Fig. 36*a*.

The magic is then formed by revolving the diagonals 180° as is shown in Fig. 36, or by interchanging the numbers represented by like-letters in Fig. 34.

Another form of natural equation square is shown in Fig. 38.

[3] A concise description of Nasik Squares is given in *Enc. Brit.*

The diagonals in this square depend on the equation $a \times b = c \times d$ (see Fig. 36*b*). The magic is made by interchanging the numbers

a			
		b	
	c		
			d

Fig. 36*a*.

a			
	b		
			c
		d	

Fig. 36*b*.

represented by like letters in Fig. 37, producing Fig. 39 and then adjusting to bring the numbers represented by the A's and D's in

		B	A
		A	B
C	D		
D	C		

Fig. 37.

42 ×	35 =	70 ×	21
×	×	×	×
2 ×	4 =	8 ×	1
=	=	=	=
6 ×	5 =	10 ×	3
×	×	×	×
14 ×	28 =	56 ×	7

Fig. 38.

42	35	1	8
2	4	21	70
28	14	10	3
5	6	56	7

Fig. 39.

$P = 11760$

6	56	7	5
35	1	8	42
4	21	70	2
14	10	3	28

Fig. 40.

Fig. 37, in one diagonal and the numbers represented by the B's and C's in the other diagonal, or in other words, shifting the left-hand column of Fig. 39 so as to make it the right-hand column,

		2	3/2	4/3	5/4
	1	2	3	4	5
6	6	12	18	24	30
7/6	7	14	21	28	35
11/7	11	22	33	44	55
13/11	13	26	39	52	65

Fig. 41.

$P = 720720$

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

Fig. 42.

1	7	13	6	11
6	11	1	7	13
7	13	6	11	1
11	1	7	13	6
13	6	11	1	7

Fig. 43.

1	14	39	24	55
18	44	5	7	26
35	13	12	33	4
22	3	28	65	6
52	30	11	2	21

Fig. 44.

and then shifting the bottom line of the square thus formed to the top. The result of these changes is shown in Fig. 40.

Fig. 41 is a fifth order natural square, and Figs. 42, 43 and 44 clearly show the method of forming the magic, which is pan-diagonal.

In the same manner Dr. Planck constructed his arithmetical Nasik squares[4] of orders $4m+2$, we can likewise construct geometric squares.

Fig. 45 shows a natural 7×7 square with the central row and column cast out. This is formed by path method into the Nasik

$P > 22 \times 10^{42}$

	2	2	2^2	2	2	
	1	2	4	16	32	64
128	2^7					
128	2^{14}					
128^2	2^{28}					
128	2^{35}					
128	2^{42}					

Fig. 45.

2^6	2^{43}	2^2	2^{48}	2^1	2^{44}
2^{35}	2^{12}	2^{39}	2^7	2^{40}	2^{11}
2^{34}	2^{15}	2^{30}	2^{20}	2^{29}	2^{16}
2^0	2^{47}	2^4	2^{42}	2^5	2^{46}
2^{41}	2^8	2^{37}	2^{13}	2^{36}	2^9
2^{28}	2^{19}	2^{32}	2^{14}	2^{33}	2^{18}

Fig. 46.

square, rearranging the columns in this order 1, 4, 32, 64, 16, 2 and the rows in this order 1, 2^7, 2^{28}, 2^{42}, 2^{35}, 2^{14} and using advance move 2, 3 and a break move -1, -1.

THE FACTORIAL METHOD.

In this method we fill two primary squares, each with n sets of any n different numbers, such that each row, column, and diagonal contains each of the n different numbers.

To avoid duplicates in the magic, the primary squares should have only one number in common, or they may not have any number in common. Also, no two numbers in one primary square should have the same ratio as two numbers in the other primary square.

This may be more clearly explained by an example. Suppose we select two sets of numbers as follows for constructing a fourth order square.

1 2 4 7
1 3 5 6

Four sets of the upper row of numbers are to fill one primary

[4] See article by Messrs. Andrews and Frierson in *The Monist*, Vol. XXII, No. 2.

square and four sets of the lower row are to fill the other. These two groups contain only one number in common, but the magic would contain duplicate numbers due to the duplicate ratios 2:4 as 3:6. Therefore $2 \times 6 = 4 \times 3$, consequently the duplicate numbers would be 12. But if we interchange the numbers 2 and 5, the fault will be corrected and the square can then be constructed without duplicate numbers.

The square in Fig. 47 is constructed with the two groups

1 2 3 4
1 5 6 7

P = 5040

1	15	24	14
12	28	3	5
21	6	10	4
20	2	7	18

Fig. 47.

P = 362880

1	10	21	32	54
28	48	9	2	15
18	3	20	42	8
30	7	16	27	4
24	36	6	5	14

Fig. 48.

A fifth order square is shown in Fig. 48 and in this case the following groups are used:

1 2 3 4 6
1 5 7 8 9

This square is pan-diagonally magic.

I will now show how a Nasik sixth order square may be made by a method derived from Dr. Planck's method of constructing Nasik squares with arithmetical series.

Fill two six-celled rectangles, each with six different numbers, the two rectangles to have no more than one number in common.

1	32	16
64	2	4

Fig. 49.

1	243	81
729	3	9

Fig. 50

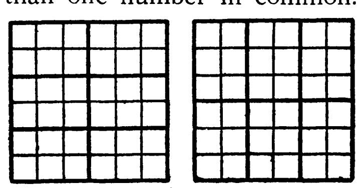

Fig. 51

The numbers in each rectangle should be arranged so that the products of its horizontal rows are equal, and the products of its vertical rows are equal.

Two of such sets of numbers that will suit the above conditions will not be found so readily as in Dr. Planck's examples above mentioned.

The two sets forming the magic rectangles in Figs. 49 and 50 are taken from the following groups:

$2^0\ 2^1\ 2^2\ 2^3\ 2^4\ 2^5\ 2^6$

$3^0\ 3^1\ 3^2\ 3^3\ 3^4\ 3^5\ 3^6$

Each group is a geometrical series of seven numbers, and in forming the rectangle, the central number in each group is omitted.

The rectangles are arranged in primary squares as shown in Fig. 51, and the two rectangles in Figs. 49 and 50 so arranged will produce the square in Fig. 52. This square is pandiagonal, 2^2-ply and 3^2-ply.[5]

P = 101,559,956,668,416.

729	192	9	46656	3	576
32	486	2592	2	7776	162
11664	12	144	2916	48	36
1	15552	81	64	243	5184
23328	6	288	1458	96	18
16	972	1296	4	3888	324

Fig. 52.

GEOMETRIC MAGIC CUBES.

I will here briefly describe the analogy between the series which may be used in constructing cubes, and those used in constructing squares.

It is obvious that an unbroken geometric series of any sort may be arranged in a cube of any order, by placing the numbers in the cube in the same progression as the numbers of an arithmetical series would be placed in forming an arithmetical cube. This may be accomplished by an extension of the method exemplified in Figs. 1 to 14 inclusive.

1	2	4
5	10	20
25	50	100

3	6	12
15	30	60
75	150	300

9	18	36
45	90	180
225	450	900

Fig. 53.

In using the Exponential La Hirian method, the same process is followed in cubes as in squares, the main difference being that three primary cubes are necessarily used.

Fig. 53 shows a natural cubic series, obtained by the Ratio

[5] A square is said to be m^2-ply when the numbers in any m^2 group of contiguous cells give a constant product in geometric squares, or a constant sum in arithmetical squares.

method. The three squares represent the three planes of the cube. The numbers 5 at the left of the first square represent the ratio between vertically adjacent cells in each of the planes. The numbers 2 above represent the ratio between horizontally adjacent cells in each of the planes, and the numbers 3 between the squares represent the ratio between adjacent cells from plane to plane.

By rearranging this series into a cube according to the path methods as in arithmetical cubes,[6] many results may be obtained, one of which is shown in Fig. 54.

A fourth order balanced or associated series is shown in Fig. 55. This series is analogous to the plane series in Fig. 30, and may

1	90	300
60	25	18
450	12	5

150	4	45
9	30	100
20	225	6

180	75	2
50	36	15
3	10	900

$P = 27000$

Fig. 54.

be transformed into a magic cube by the following well-known method:

Ratios: horizontal 2, $\frac{5}{2}$, 2; vertical 3, $\frac{4}{3}$, 3; from plane to plane 7, $\frac{9}{7}$, 7.

1	2	5	10
3	6	15	30
4	8	20	40
12	24	60	120

7	14	35	70
21	42	105	210
28	56	140	280
84	168	420	840

9	18	45	90
27	54	135	270
36	72	180	360
108	216	540	1080

63	126	315	630
189	378	945	1890
252	504	1260	2520
756	1512	3780	7560

Fig. 55.

Interchange the numbers in all associated pairs of cells which are inclosed in circles, producing the result shown in Fig. 56.

$P = 57{,}153{,}600$

7560	2	5	756
3	1260	504	30
4	945	378	40
630	24	60	63

7	540	216	70
360	42	105	36
270	56	140	27
84	45	18	840

9	420	168	90
280	54	135	28
210	72	180	21
108	35	14	1080

120	126	315	12
189	20	8	1890
252	15	6	2520
10	1512	3780	1

Fig. 56.

The possibilities in using the Factorial method in constructing cubes, has not been investigated by the writer.

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[6] See *Magic Squares and Cubes* by W. S. Andrews.